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Andrew John Anderson
Iowa State University

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REVERSE TRANSIENT RESPONSE IN A P I N DIODE

by

Andrew John Anderson

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Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1963

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INTRODUCTION

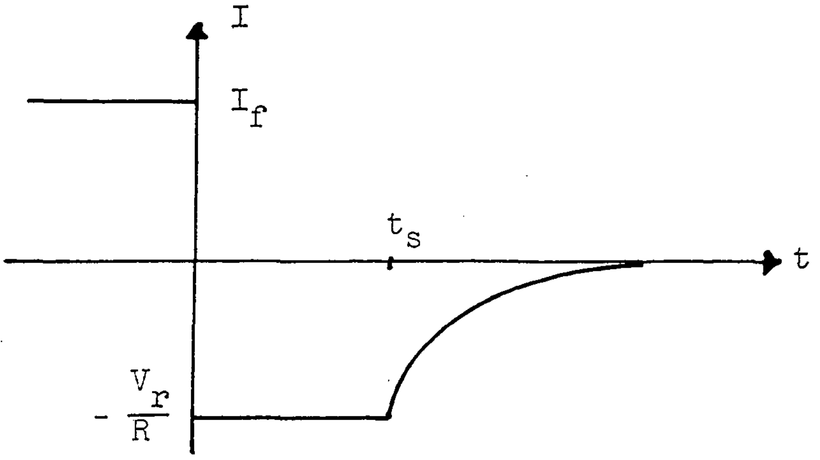
A reverse transient current possessing a wave form similar to that shown in Figure 1a is observed when a semiconductor junction diode is incorporated in a circuit similar to that shown in Figure 1b. The diode is initially in a state of forward conduction with the switch in position A. At time t = 0, the circuit is switched to position B and the diode is in series with a resistance R and a voltage source V_r. Figure 1a is a typical waveform of the current observed under these conditions.

Due to the growing need for more and more high speed semiconductor switches, the need for an analysis of this transient behavior in various types of junction diodes has become of significant importance in recent years.

The type of junction diode to be analyzed in this thesis is the long PIN diode. It consists of a thin, heavily doped P region, a long intrinsic region, and a thin, heavily doped N region. The method to be employed is to analyze the transport of holes and electrons from the intrinsic region. These carriers, which were injected into the intrinsic region during forward conduction, will diffuse out across the junction boundaries and give rise to a reverse transient current similar to that shown in Figure 1a.

Figure 1a. The waveform of the transient current.

Figure 1b. The circuit giving rise to the transient phenomena


I
I_f
t_s
t
$-\frac{V_r}{R}$

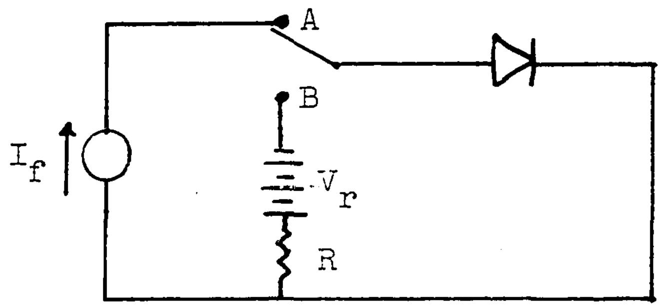

A
B
I_f
V_r
R

Table 1. Definition of symbols used

t_1	real time
t	normalized time $t = t_1/\tau$
τ	bulk lifetime of minority carriers
D_n	electron diffusion constant
D_p	hole diffusion constant
μ_n	electron mobility
μ_p	hole mobility
b	mobility ratio $b = \mu_p/\mu_n$
L	high level diffusion length $L^2 = \frac{2D_n\tau}{1+1/b}$
z	real space variable
x	normalized space variable $x = z/L$
I_f	forward current
I_r	reverse recovery current
i_n	electron current
i_p	hole current
q	electronic charge
V_a	potential change at i-p junction
V_d	potential change at i-n junction
k	Boltzmann constant
ε	permitivity
E	electric field
T	temperature
s	Laplace transform variable

Table 1. (Continued)

w	$w = \sqrt{1+s}$
n	density of electrons / cm^3
p	density of holes / cm^3
n_i	intrinsic carrier density / cm^3
ρ	charge density
d	normalized intrinsic region length
g	generation rate of intrinsic carriers
V_f	forward potential at p-n junction

PREVIOUS WORK

There has been a considerable amount of previous work done in this general area. A number of papers have been written concerning the reverse transient response in the p-n junction diode (1, 2, 3, 4, 5, 6). Also, several papers have been written concerning the steady state forward conduction problem in the PIN diode (7, 8, 9, 10). These papers were of great help in developing an insight to the particular problem analyzed in this thesis.

Reverse Transient Response

In general, all work concerning the transient response in a junction diode has been performed by analyzing the excess minority carriers injected across the junction boundary during forward conduction. The common method of analysis was to determine the reverse transient current in terms of the transport of these excess minority carriers back across the junction boundary.

Work has been done on both the step junction and the graded base diode. The difference in the two types results from the fact that in the step junction, the internal field effect is neglected (1, 2, 3, 4, 6). In the graded base diode, the field is assumed to be of constant magnitude (4, 5, 6). This previous research on the p-n junction diode has been performed under various combinations of the following assumptions:

1. One side of the p-n junction is of considerably higher conductivity so that only one type of carrier on one side of the junction need be considered (1, 2, 3, 4, 5, 6).

2. The internal field effect is negligible (1, 2, 3, 4, 6).

3. The electric field is of constant magnitude (4, 5, 6).

4. The bulk lifetime of minority carriers is constant (1, 2, 3, 4, 5, 6).

5. Recombination is linear (1, 2, 3, 4, 6).

6. Recombination is neglected (5).

It should now be of value to briefly describe the methods and results obtained in one of the more recent papers in this area in order to develop an insight to this type of problem (6).

The basic equation which defines the current in a semiconductor is given by Shockley (11).

$$i_n = q(n \mu_n E + \frac{D_n}{L} \frac{\partial n}{\partial x}) \qquad 2\text{-}1$$

During steady state forward conduction, the excess minority carrier distribution takes on the form

$$n(x) = \frac{n_i^2}{N_A} e^{-x} e^{\frac{qVf}{kT}} \qquad 2\text{-}2$$

under the assumption of no internal electric field. This distribution is to be the distribution at time t = 0 when the reverse transient begins.

The equation describing the time rate of change of the distribution is

$$\frac{\partial n}{\partial t} = \frac{-n}{\tau} + \frac{\mathrm{div}}{q}\, i_n, \tag{2-3}$$

under the assumption of a linear recombination law and no internal field. Combining Equations 2-1 and 2-3 yields

$$\frac{\partial n}{\partial t} = + \frac{\partial^2 n}{\partial x^2} - n. \tag{2-4}$$

The boundary condition at x = 0 is given by Equation 2-1 evaluated at x = 0. The second boundary condition is $n(\infty, t) = 0$ in the analysis of a semi-infinite diode.

Since the reverse recovery current is limited by the resistance R in the circuit of Figure 1b, the gradient of the distribution function must be constant at the boundary x = 0 during the constant current part of the recovery period. This is because the junction must remain with a junction potential between zero and the forward bias potential due to the fact that $n(0, t)$ is greater than zero for $t < t_s$. A reverse bias condition, which would create a drift field and hence a drift current across the boundary, cannot occur until the gradient of the distribution function drops below a value that would support the current V_r/R. This situation cannot occur until $n(0, t) \approx 0$. Therefore the boundary condition for the constant current part of the recovery period is

$$I_r = q \frac{D_n}{L} \frac{\partial n}{\partial x} \Big|_{x = 0} = \frac{-V_r}{R}. \tag{2-5}$$

The solution of Equation 2-4 subject to the boundary conditions $n(\infty, t) = 0$ and Equation 2-5 with the initial condition given by Equation 2-2 is

$$n(x, t) = \frac{I_r - I_f}{2}\left[e^{-x} \operatorname{erfc}\left(\frac{x}{2\sqrt{t}} - \sqrt{t}\right) - e^{x} \operatorname{erfc}\left(\frac{x}{2\sqrt{t}} + \sqrt{t}\right)\right] + I_f\, e^{-x} . \qquad 2\text{-}6$$

The termination of the constant current period at time t_s is given by the time when $n(0, t)$ reaches zero. Solving Equation 2-6 for this time yields

$$\operatorname{erf} \sqrt{t_s} = \frac{I_f}{|I_r| + |I_f|} . \qquad 2\text{-}7$$

A plot of Equation 2-6 for several values of time is shown in Figures 2a to 2d (6).

For the nonconstant current part of the recovery period, Equation 2-4 is again solved, but with new initial and boundary conditions.

$$n(0, t) = 0 \qquad 2\text{-}8$$

$$n(\infty, t) = 0 \qquad 2\text{-}9$$

The initial condition is given by Equation 2-6 evaluated at time $t = t_s$. The solution of the above system yields a long, complicated solution due to the complex initial condition. The current itself is found by evaluating Equation 2-1 at the boundary $x = 0$, and it possesses approximately the waveform shown in Figure 1a.

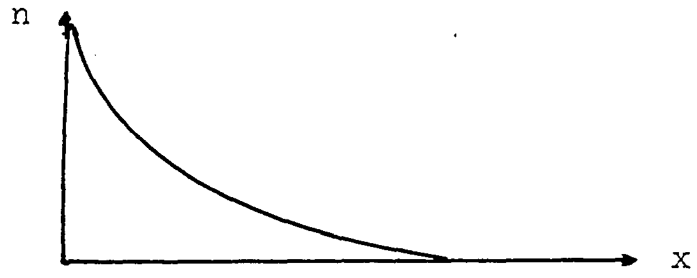
Figure 2a. The minority carrier distribution in the p-n junction for $t = 0$

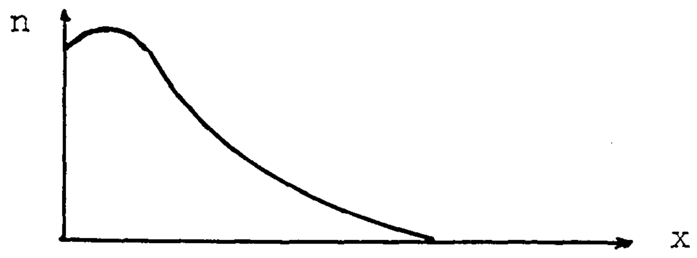
Figure 2b. The minority carrier distribution in the p-n junction for $t > 0$

Figure 2c. The minority carrier distribution in the p-n junction for $t > 0$

Figure 2d. The minority carrier distribution in the p-n junction for $t = t_s$

Figure 2e. The minority carrier distribution in the p-n junction for $t > t_s$


n
x


n
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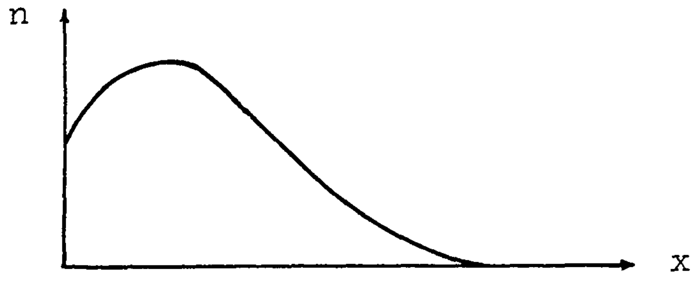
n
x


n
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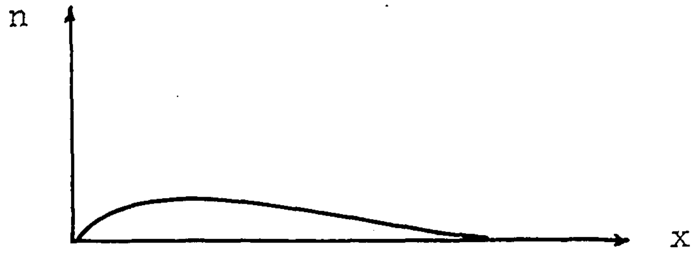

n
x

The Forward Characteristic of the PIN Diode

All previous work concerned only the carrier distribution in the PIN diode during forward conduction.

The differential equations describing the carrier distributions within the PIN diode, which will be derived in detail in the next chapter, are unfortunately non linear equations. The non linearity results from the fact that the internal field given by Maxwell's equation is dependent upon the distributions themselves.

$$\text{div } E = \frac{q(p-n)}{\varepsilon} \qquad 2\text{-}10$$

The method employed to obtain a set of linear differential equations which can be solved analytically is to note that the field is small in magnitude and therefore the hole distribution must be approximately equal to the electron distribution (7, 8, 9, 10). This point is discussed in more detail in Chapter 3 and in Appendix A.

The results obtained in previous work on the PIN junction will not be described here, but will be derived in detail in the next chapter.

Table 2. A list of special constants

$$A_1 = \frac{(b+1)\cosh d}{[(b+1)^2(\cosh d)^4 - (b-1)^2(\sinh d)^4]^{1/2}}\; n_i \; e^{q\frac{(V_a+V_d)}{2kt}}$$

$$A_2 = \frac{-(b-1)\sinh d}{[(b+1)^2(\cosh d)^4 - (b-1)^2(\sinh d)^4]^{1/2}}\; n_i \; e^{q\frac{(V_a+V_d)}{2kt}}$$

$$A_3 = A_1 \sinh d$$

$$A_4 = A_2 \cosh d$$

$$A_5 = A_1 \cosh d$$

$$A_6 = A_2 \sinh d$$

$$I_1 = \frac{L\, I_r}{2qD}$$

$$I_2 = \frac{L\, I_r\, b}{2qD}$$

ANALYSIS OF THE JUNCTION

Introduction

A PIN diode, often called a majority carrier contact diode is represented schematically in Figure 3a. It consists of a relatively thin, heavily doped p layer, an intrinsic region of arbitrary thickness, and a thin, heavily doped n layer. There are actually two junctions in the device, one at $z = -d$ and one at $z = d$. Assuming that the p and n doping levels are approximately the same, there are two thin space charge regions of thickness Δd occurring at d and -d respectively. The approximate potential diagram for this device under the steady state condition of no applied bias voltage is shown in Figure 3b (7, 8).

To determine the distribution of carriers within the diode during a state of forward conduction, we must next apply some elementary transport theory.

The time rate of change of carrier concentration within the intrinsic region of the diode is given by the following equation:

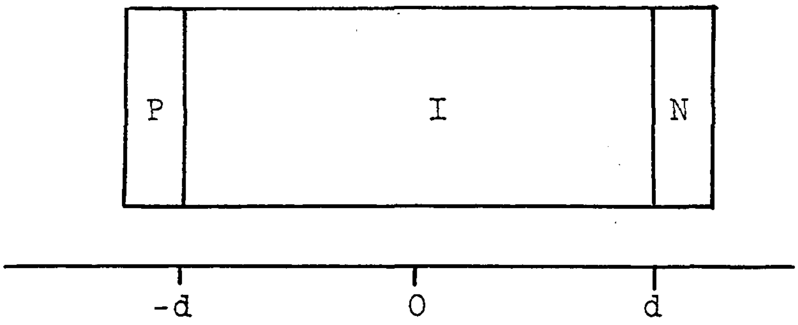
$$\begin{array}{l}\text{change in}\\ \text{n per unit}\\ \text{time}\end{array} = \begin{array}{l}\text{change in n}\\ \text{due to recom-}\\ \text{bination and}\\ \text{generation}\end{array} \pm \frac{\text{divergence of } \vec{i}}{q} . \qquad \text{3-1}$$

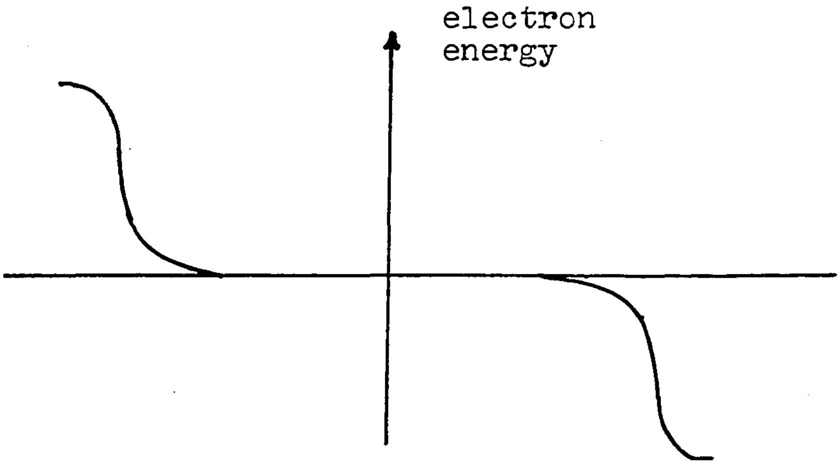
By writing Equation 3-1 in symbolic form, we obtain

$$\frac{\partial n}{\partial t_1} = g - R + \frac{\text{div}}{q}\vec{i}_n \qquad \text{3-2}$$

Figure 3a. The PIN junction diode

Figure 3b. The potential diagram for the PIN junction diode


P
I
N
-d
0
d


electron
energy

$$\frac{\partial p}{\partial t_1} = g - R - \frac{\operatorname{div}}{q} \vec{i}_p \quad . \tag{3-3}$$

From Shockley (11), we obtain the equations defining the electron and hole currents in a semiconductor.

$$\vec{i}_n = q\mu_n n\vec{E} + qD_n \operatorname{gr\vec{a}d} n \tag{3-4}$$

$$\vec{i}_p = q\mu_p p\vec{E} - qD_p \operatorname{gr\vec{a}d} p \quad . \tag{3-5}$$

By inserting Equations 3-4 and 3-5 into Equations 3-2 and 3-3, we obtain the differential equations governing the electron and hole distributions.

$$\frac{\partial n}{\partial t_1} = g - R + \operatorname{div} n\vec{E} + D_n \nabla^2 n \tag{3-6}$$

$$\frac{\partial p}{\partial t_1} = g - R - \operatorname{div} p\vec{E} + D_p \nabla^2 p \quad . \tag{3-7}$$

Before proceeding further in the development of the differential equations, it will be convenient to state the assumptions to be used in the analysis of this model.

Assumptions

1. The thermal generation of carriers within the diode shall be neglected. Since this is an analysis of a transient phenomena caused by the presence of a large number of injected carriers, it shall be assumed that the effect of the relatively small number of thermally generated carriers is very small.

2. A linear recombination law shall be assumed. Although experiment indicates that this is probably not true in silicon

(8), it has been a widely accepted practice since it allows us to obtain a linear differential equation.

3. Since the device under consideration is several diffusion lengths long, it shall be assumed that the forward current is due entirely to recombination of carriers within the intrinsic region (7).

4. The intrinsic region is much longer than the space charge regions shown in Figure 3a.

5. The intrinsic region is at all times a quasi-neutral region containing no net charge (7, 8, 9, 10). This assumption will be discussed in more detail later in this chapter, and its validity will be discussed in Appendix A.

6. The bulk lifetime of minority carriers within the intrinsic region shall be assumed to be constant.

7. The diode shall be assumed to be in a steady state forward conduction condition prior to the occurrence of the recovery transient.

8. Lateral variations across the diode shall be considered negligible so that the junction may be represented by a one dimensional model.

Differential Equations

The basic equations governing the conduction process are those given by Equations 3-6 and 3-7. The effects of the assumptions shall be applied to these equations to reduce them to a form that can be readily solved.

The effect of assumption 1 is to set g equal to zero in Equations 3-6 and 3-7.

The effect of assumptions 2 and 6 is to define the recombination function R in terms of n, p, and τ as in Equations 3-8 and 3-9.

$$R_n = \frac{n}{\tau}, \qquad 3\text{-}8$$

$$R_p = \frac{p}{\tau}. \qquad 3\text{-}9$$

Using Equations 3-8 and 3-9 in Equations 3-6 and 3-7, we may now write the two differential equations as follows for the one dimensional case.

$$\frac{\partial n}{\partial t_1} = -\frac{n}{\tau} + \mu_n \frac{\partial}{\partial z}(nE) + D_n \frac{\partial^2 n}{\partial z^2} \qquad 3\text{-}10$$

$$\frac{\partial p}{\partial t_1} = -\frac{p}{\tau} - \mu_p \frac{\partial}{\partial z}(pE) + D_p \frac{\partial^2 p}{\partial z^2} \qquad 3\text{-}11$$

Equation 3-11 may now be rewritten in terms of μ_n, D_n, and b, by means of the Einstein relationship.

$$\frac{1}{b}\frac{\partial p}{\partial t_1} = -\frac{1}{b}\frac{p}{\tau} - \mu_n \frac{\partial}{\partial z}(pE) + D_n \frac{\partial^2 n}{\partial z^2} \qquad 3\text{-}12$$

Next, Equations 3-10 and 3-12 are added together.

$$\frac{\partial n}{\partial t_1} + \frac{1}{b}\frac{\partial p}{\partial t_1} = \frac{-n}{\tau} - \left(\frac{1}{b}\right)\frac{p}{\tau} - \mu_n \frac{\partial}{\partial z}[(p-n)E] + D_n \frac{\partial^2}{\partial z^2}(p+n) \qquad 3\text{-}13$$

Now, the effect of assumption 5, that of quasi-neutrality within the intrinsic region must be considered. The first effect of this assumption is to require the hole flow into or away from the intrinsic region be equal the electron flow into or away from the intrinsic region. This prevents the formation of a space charge field across the junction boundaries. The internal field occurring within the intrinsic region is that given by Maxwell's equation.

$$\text{div } E = \frac{\rho}{\varepsilon} \qquad 3\text{-}14$$

In terms of the charge densities involved, this becomes

$$E(x) = \frac{q}{\varepsilon} \int_{-d}^{x} [p(y) - n(y)]dy \qquad 3\text{-}15$$

The second effect of this assumption, which is discussed in detail in Appendix A, is to imply that the field E(x) is small in magnitude and can be approximated by zero if it is specified that the hole concentration must by approximately equal to the electron concentration. In other words, since the concentrations will distribute themselves so as to minimize the electric field, this minimum will occur when p is approximately equal to n. As is shown in Appendix A, this error is approximately 1 part in 10^6 during forward conduction and increases to about 1 part in 100 at the end of the constant current recovery period. Applying this result to Equation 3-13, we obtain

$$(1 + \frac{1}{b}) \frac{\partial n}{\partial t_1} = -(1 + \frac{1}{b}) \frac{n}{\tau} + 2D_n \frac{\partial^2 n}{\partial z^2}. \qquad 3\text{-}16$$

Equation 3-16 is now divided through by $(1 + \frac{1}{b})/\tau$ yielding

$$\tau \frac{\partial n}{\partial t_1} = \frac{2D_n\tau}{(1+\frac{1}{b})} \frac{\partial^2 n}{\partial z^2} - n \, . \qquad 3\text{-}17$$

Since a diffusion length is defined as $(D\tau)^{1/2}$, it is convenient to define a high level diffusion length L, as described in Table 1. Equation 3-17 then becomes

$$\tau \frac{\partial n}{\partial t_1} = L^2 \frac{\partial^2 n}{\partial z^2} - n \, . \qquad 3\text{-}18$$

By now making a substitution to a normalized set of variables as listed in Table 1, Equation 3-18 becomes

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} - n \qquad 3\text{-}19$$

and

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} - p \, . \qquad 3\text{-}20$$

Steady State Forward Distribution

To obtain the steady state forward carrier distribution, either Equation 3-19 or 3-20 must be solved. To obtain the boundary conditions necessary for a unique solution, the currents across the boundaries of the intrinsic region must be considered.

Equations 3-21 and 3-22 relate the change in potential at the boundaries to the carrier concentrations.

$$n(-d) = p(-d) = n_i \exp \frac{qV_a}{kT} \qquad 3\text{-}21$$

$$n(d) = p(d) = n_i \exp \frac{qV_d}{kT} \qquad 3\text{-}22$$

The requirement of quasi-neutrality dictates that the concentration gradients at the boundaries must differ by the mobility ratio b (7). This result is obtained by assuming that the p-i junction is a perfect hole emitter and the i-n junction is a perfect electron emitter. Further, the charge flow across the p-i junction must be due almost entirely to holes and that at the i-n junction due almost entirely to electrons. This requirement is met so long as the boundary values given by Equations 3-21 and 3-22 are much less in magnitude than the impurity levels in the n and p layers, which is true for nearly all reasonable current values (10).

$$\left.\frac{\partial n}{\partial x}\right|_{x=-d} = -b \left.\frac{\partial n}{\partial x}\right|_{x=d} \qquad 3\text{-}23$$

To obtain the steady state forward distribution, Equation 3-19 is used along with boundary condition 3-23 and the fact that $\frac{\partial n}{\partial t} = 0$.

$$\frac{\partial^2 n}{\partial x^2} = n \qquad 3\text{-}24$$

By solving Equation 3-24 and substituting Equations 3-21 and 3-22, the following solution is obtained for the steady state carrier distribution.

$$n(x) = p(x) = \frac{[(b+1)\cosh d \cosh x - (b-1)\sinh d \sinh x]}{[(b+1)^2(\cosh d)^4 - (b-1)^2(\sinh d)^4]^{1/2}} . \qquad 3\text{-}25$$

$$\cdot\ n_i \exp \frac{q(V_a+V_d)}{2kT}$$

This solution is carried out in detail in Appendix B. The result is identical to that derived by R. N. Hall (7).

This is performed by integrating the recombination function R over the entire intrinsic region.

$$I_f = qL\int_{-d}^{d} R\,dx = \frac{qL}{\tau}\int_{-d}^{d} [n(y) + p(y)]dy \qquad 3\text{-}26$$

$$= \frac{\frac{2D}{L}q(b+1)\cosh d \sinh d\; n_i \exp \frac{q(V_aV_d)}{2kT}}{[(b+1)^2(\cosh d)^4 - (b-1)^2(\sinh d)^4]^{1/2}} \qquad 3\text{-}27$$

Constant Current Recovery Solution

As in the forward steady state case, the assumption of quasi-neutrality in the intrinsic region is again applied. During the constant current period, the current is limited by the resistance R in the circuit of Figure 1. Since a diffusion current due to a concentration gradient across the boundaries can itself cause an arbitrarily large recovery current, the junction cannot become reverse biased until the concentrations reach zero at the boundaries. A reverse bias condition would lead to a drift current superimposed on the diffusion current. But since the total current is limited to V_r/R, this drift current component cannot exist, and hence the junction must be in a state of approximately no bias.

Because there exists a large potential barrier for holes at x = d and a large potential barrier for electrons at x = -d, the recovery current must be composed of a hole flow from the boundary at x = -d and an electron flow from the boundary at $x = d$. To maintain total charge neutrality in the intrinsic region these two currents must be equal. Also, because of the requirement that p is approximately equal to n, the equations imply that there is also a hole diffusion current across the boundary $x = d$ and an electron current across the boundary $x = -d$. However, there exists a potential barrier retarding the flow of holes across $x = d$ and a second potential barrier that retards the flow of electrons across the boundary $x = -d$. Since a negligibly small number of these carriers cross the boundaries in question during forward conduction (7), it is assumed that this number of carriers is even more negligible during the recovery case because the potential barriers are at least as large as during forward conduction. The small internal field creates a drift current which transports these carriers across the entire intrinsic region toward the boundaries across which they can diffuse. As was pointed out in Appendix A, the assumption that $p = n$ is quite good and therefore the distributions possess almost exactly the same shape, even at the boundaries. By the argument given in the previous paragraph, the electron current at the boundary $x = -d$ must be zero and the hole current must be equal to the total recovery current I_r. Therefore, there must be an electron drift current component at the boundary which exactly cancels the diffusion current component which is specified by the electron gradient at the boundary. The electric field which creates this electron drift current also creates a hole drift current at the boundary. Since the electron drift and diffusion currents are opposite in direction, the hole drift and diffusion currents must be in the same direction, and thus they must add up to the total recovery current I_r. Further, the hole drift and diffusion currents must be of exactly the same magnitude because of the Einstein relationship and the fact that drift and diffusion electron currents are required to be of the same magnitude. Because of this the diffusion currents measured at each boundary are exactly one half of the total recovery current.

From Equations 3-4 and 3-5, the boundary conditions for the constant current period are obtained.

$$\frac{bqD}{L}\frac{\partial n}{\partial x}\Big|_{x=d} = \frac{I_r}{2} \qquad 3\text{-}28$$

$$\frac{-qD}{L}\frac{\partial n}{\partial x}\Big|_{x=-d} = \frac{I_r}{2} \qquad 3\text{-}29$$

To obtain the distribution functions for the carriers during this period, Equation 3-19 must be solved subject to the boundary conditions given by Equations 3-28 and 3-29. The initial condition, the distribution at time $t = 0$, is given by the steady state forward distribution represented by Equation 3-25. Before proceeding with the solution of the equation, it is convenient to define a set of constants to simplify the appearance of the solution, as is done in Table 2.

In terms of the constants defined in Table 2, the system to be solved now becomes

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} - n \qquad 3\text{-}29$$

$$\frac{\partial n}{\partial x}\bigg|_{x=-d} = I_1 \qquad 3\text{-}30$$

$$\frac{\partial n}{\partial x}\bigg|_{x=d} = I_2 \qquad 3\text{-}31$$

$$n(x,\ 0) = A_1 \cosh x + A_2 \sinh x\ . \qquad 3\text{-}32$$

The detailed solution of this system is carried out in Appendix C, yielding the following result for the carrier concentrations within the intrinsic region.

$$\begin{aligned}
n(x,\ t) = p(x,\ t) = {} & A_1 \cosh x + A_2 \sinh x - \frac{(I_1+A_3+A_4)}{2} \\
& e^{-(d-x)} \operatorname{erfc}\left[\frac{d-x}{2\sqrt{t}} - \sqrt{t}\right] - e^{d-x} \operatorname{erfc}\left[\frac{d-x}{2\sqrt{t}} + \sqrt{t}\right] \\
& + \sum_{n=0}^{\infty} \Bigg[\frac{(I_2+A_4-A_3)}{2} \Big[e^{-[4n+3)d-x]} \operatorname{erfc}\left[\frac{(4n+3)d-x}{2\sqrt{t}} - \sqrt{t}\right] - e^{[(4n+3)d-x]} \\
& \operatorname{erfc}\left[\frac{(4n+3)d-x}{2\sqrt{t}} + \sqrt{t}\right] \\
& + e^{-[(4n+1)d+x]} \operatorname{erfc}\left[\frac{(4n+1)d+x}{2\sqrt{t}} - \sqrt{t}\right] - e^{[(4n+1)d+x]} \operatorname{erfc}\left[\frac{(4n+1)d+x}{2\sqrt{t}} + \sqrt{t}\right] \Bigg] \\
& - \frac{(I_1+A_3+A_4)}{2} \Big[e^{-[(4n+5)d-x]} \operatorname{erfc}\Big[\frac{(4n+5)d-x}{2\sqrt{t}} - \sqrt{t} \\
& - e^{[(4n+5)d-x]} \operatorname{erfc}\left[\frac{(4n+5)d-x}{2\sqrt{t}} + \sqrt{t}\right]
\end{aligned}$$

$$+e^{-[(4n+3)d+x]}\operatorname{erfc}\left[\frac{(4n+3)d+x}{2\sqrt{t}}-\sqrt{t}\right]-e^{[(4n+3)d+x]}\operatorname{erfc}\left[\frac{(4n+3)d+x}{2\sqrt{t}}+\sqrt{t}\right]\Bigg]\Bigg] \qquad 3\text{-}33$$

A plot of $n(x, t)$ for several values of t is displayed in Figure 4. These curves were plotted by a computer evaluation of Equation 3-33. The complementary error functions were evaluated by expanding them in a Taylor series (12).

Nonconstant Current Recovery Solution

As in the previous section Equation 3-19 is solved to determine the carrier distribution functions for this period.

The boundary conditions applying to the non constant current phase are those conditions specifying the termination of the constant current phase.

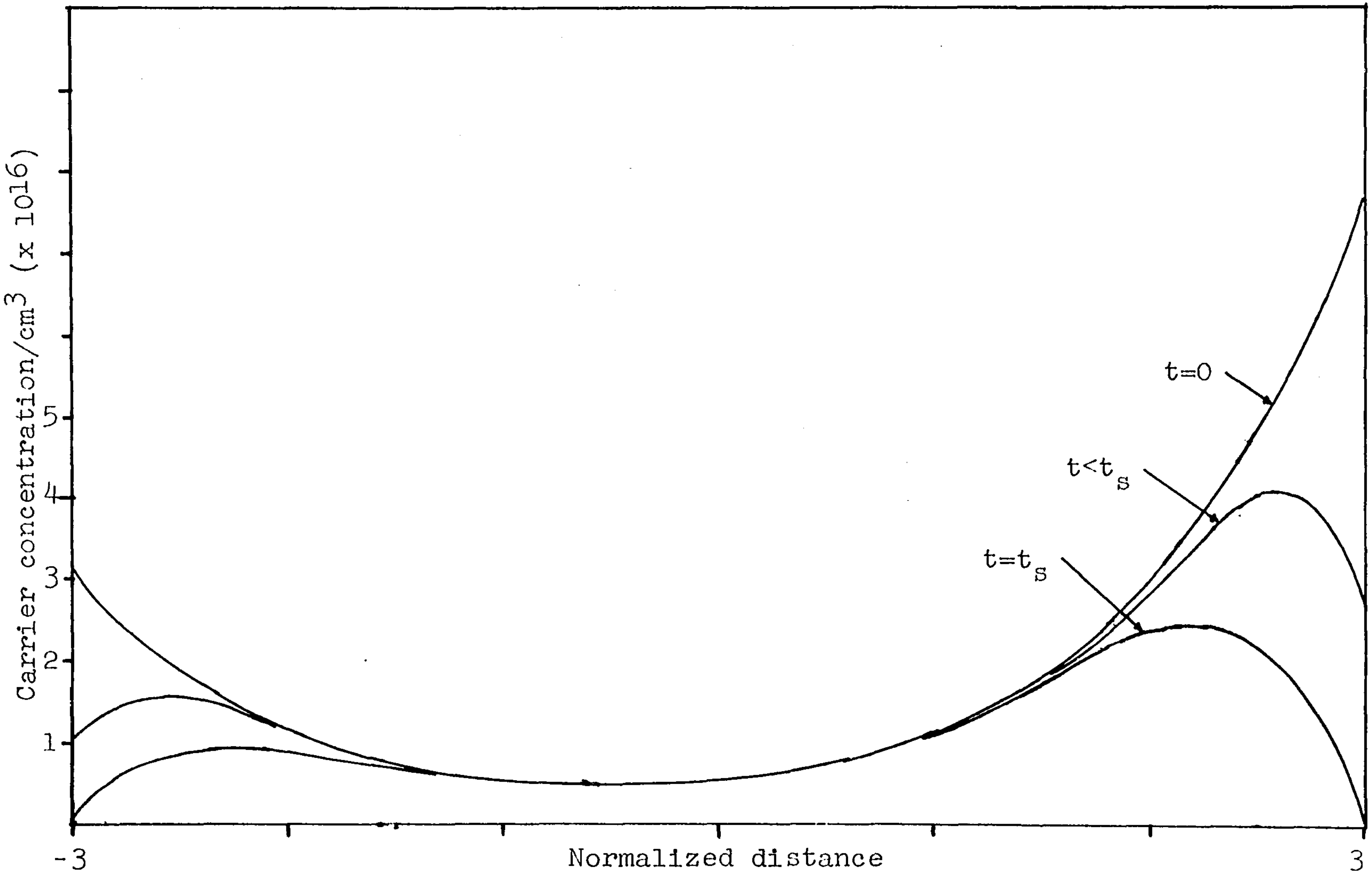
$$n(d, t) = 0 \qquad 3\text{-}34$$

$$n(-d, t) = 0 \qquad 3\text{-}35$$

The initial condition should be given by Equation 3-33 evaluated at time $t = t_s$. It is impractical to solve the boundary value problem described above due to the extremely complicated initial condition given by Equation 3-33. Another alternative is to use Equation 3-32 for the initial condition in Equation 3-19 (5). The solution of this boundary value problem is carried out in Appendix D.

In this phase of the problem, the current is the primary

Figure 4. A plot of the distribution functions during the constant current part of the recovery time


Carrier concentration/cm^3 (x 10^{16})
1
2
3
4
5
-3
Normalized distance
3
t=0
t<t$_s$
t=t$_s$

interest. This may be obtained from the carrier distribution $n(x, t)$ by applying Equations 3-4 and 3-5 at the boundary. This results in the current expression given by 3-36.

$$I_r = \frac{2qD}{L} \frac{\partial n}{\partial x}\Big|_{x=d} \qquad 3\text{-}36$$

By taking the first derivative with respect to x of the distribution function derived in Appendix D and evaluating at $x = d$, a relatively simple expression for the current results.

$$I_r = \frac{2qD}{L} \left[A_1 \sinh d = A_2 \cosh d - (A_5+A_6)\left(\mathrm{erfc}\ t\ \frac{e^{-t}}{\sqrt{\pi t}}\right)\right] \qquad 3\text{-}37$$

For very small values of t, this expression does not describe the physical problem because the current I_r is much larger than the current V_r/R which was observed during the constant current phase. Equation 3-37 is evaluated at time $t = t_2$ which is the time when I_r is equal to V_r/R. This time t_2, is the minimum value for t for which Equation 3-37 is valid. To demonstrate that the approximation involved in using Equation 3-33 for the initial condition, the distribution function for the non-constant current phase, Equation D-10, is evaluated at time $t = t_2$ and is compared to the actual initial condition, Equation 3-37 evaluated at time $t = t_s$. A plot of these two distribution functions is given in Figure 5.

Thus if $I_r(t)$ as given by Equation 3-37 is displaced by an amount $t_s - t_2$, it represents the non constant recovery current to a good approximation. A plot of Equation 3-37 and the total recovery current are displayed in Figure 6.

Figure 5. A comparison of the actual and approximate initial conditions for the non-constant current phase

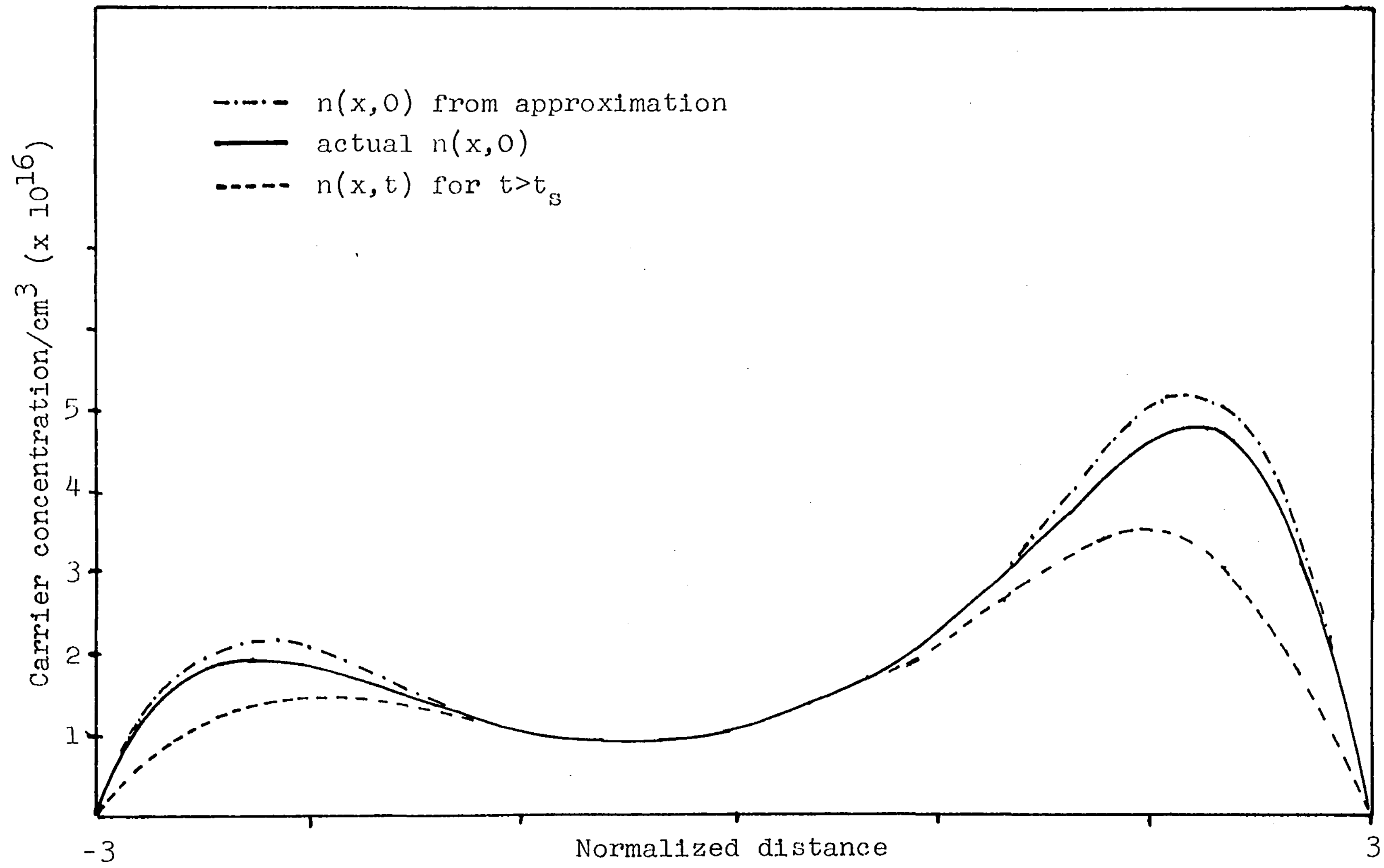
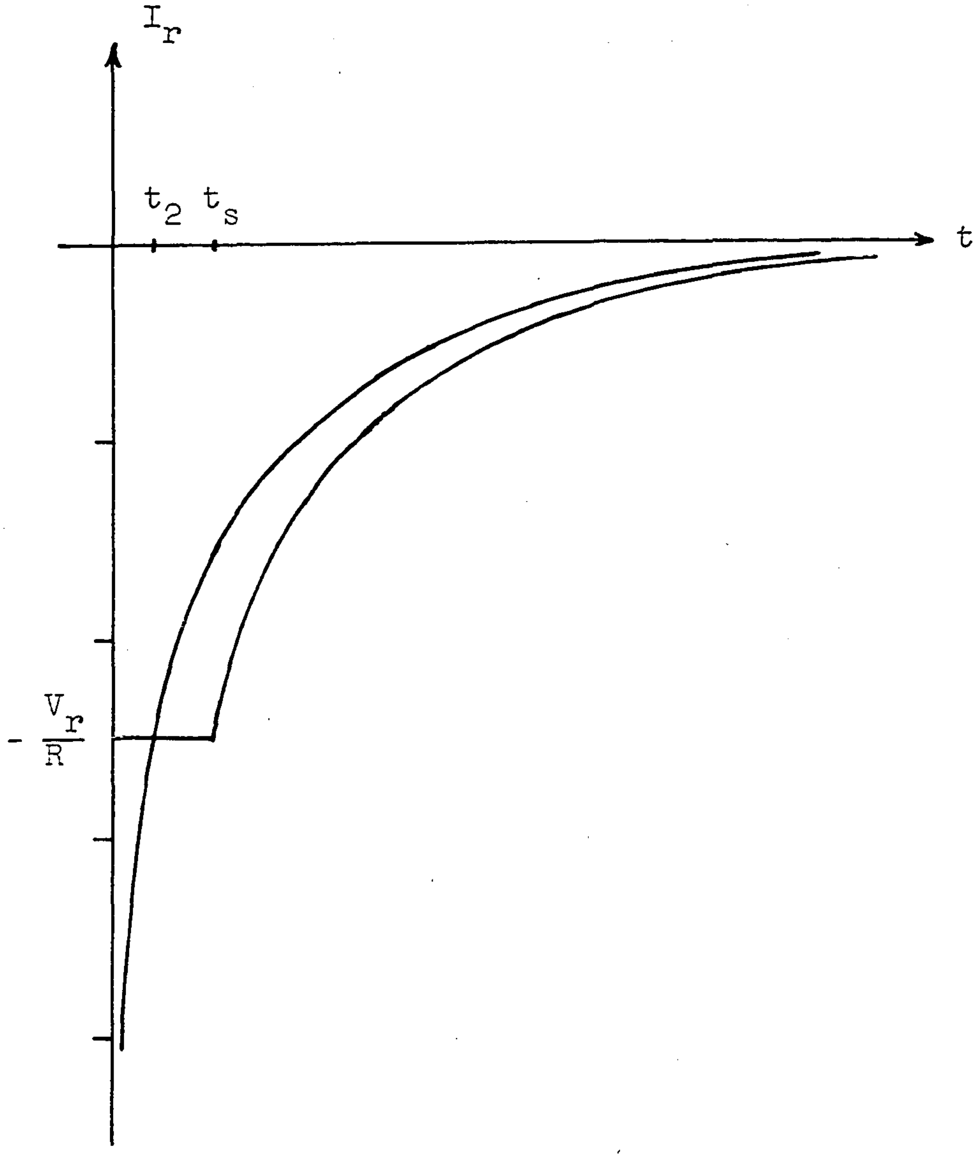

n(x,0) from approximation
actual n(x,0)
n(x,t) for t>t$_s$
Carrier concentration/cm^3 (x 10^{16})
5
4
3
2
1
-3
Normalized distance
3

Figure 6. The non-constant recovery current


I_r
t_2
t_s
t
$-\frac{V_r}{R}$

RESULTS OF THE ANALYSIS

The first result to be obtained from the equations derived in the previous section is to obtain an expression for the storage time t_s. It is convenient to obtain this expression in terms of the forward current I_f and the reverse current I_r rather than the diode parameters A_1, A_2, etc. This may be done by using the forward current expression given by Equation 3-27 to evaluate the diode parameters.

$$A_1 \sinh d = \frac{LI_f}{2qD} \qquad 4\text{-}1$$

$$A_2 \cosh d = \frac{LI_f}{2qD}\frac{(b-1)}{(b+1)} \qquad 4\text{-}2$$

$$A_5 = \frac{LI_f}{2qD} \qquad 4\text{-}3$$

$$A_6 = -\frac{LI_f}{2qD}\frac{(b-1)}{(b+1)} \qquad 4\text{-}4$$

The storage time t_s is the time required for the distribution function $n(x, t)$ as given by Equation 3-33 to reach zero at the boundaries. By restricting the recovery current to values approximately on the order of magnitude of I_f so that t_s will be less than one, a relatively simple equation may be derived for the storage time. The only other approximation involved is to neglect all terms involving the complementary error function for which the arguments are greater than two. This results in

$$\operatorname{erfc}(-\sqrt{t_s}) - \operatorname{erfc}\sqrt{t_s} = \frac{2\, A_1 \cosh d + A_2 \sinh d}{I_1 + A_5 + A_6} \qquad 4\text{-}5$$

which yields

$$\operatorname{erf}\sqrt{t_s} = \frac{I_f[\coth d - \frac{(b-1)}{(b+1)}\tanh d].}{I_r + I_f\left[1 - \frac{(b-1)}{(b+1)}\right]} \qquad 4\text{-}6$$

By utilizing the fact that d is large and thus coth d is nearly equal to tanh d, we have

$$\operatorname{erf}\sqrt{t_s} = \frac{I_f\left(\frac{2}{b+1}\right)}{I_r + I_f\left(\frac{2}{b+1}\right)} . \qquad 4\text{-}7$$

A plot of the normalized storage time t_s versus the ratio of I_r to I_f is shown in Figure 7.

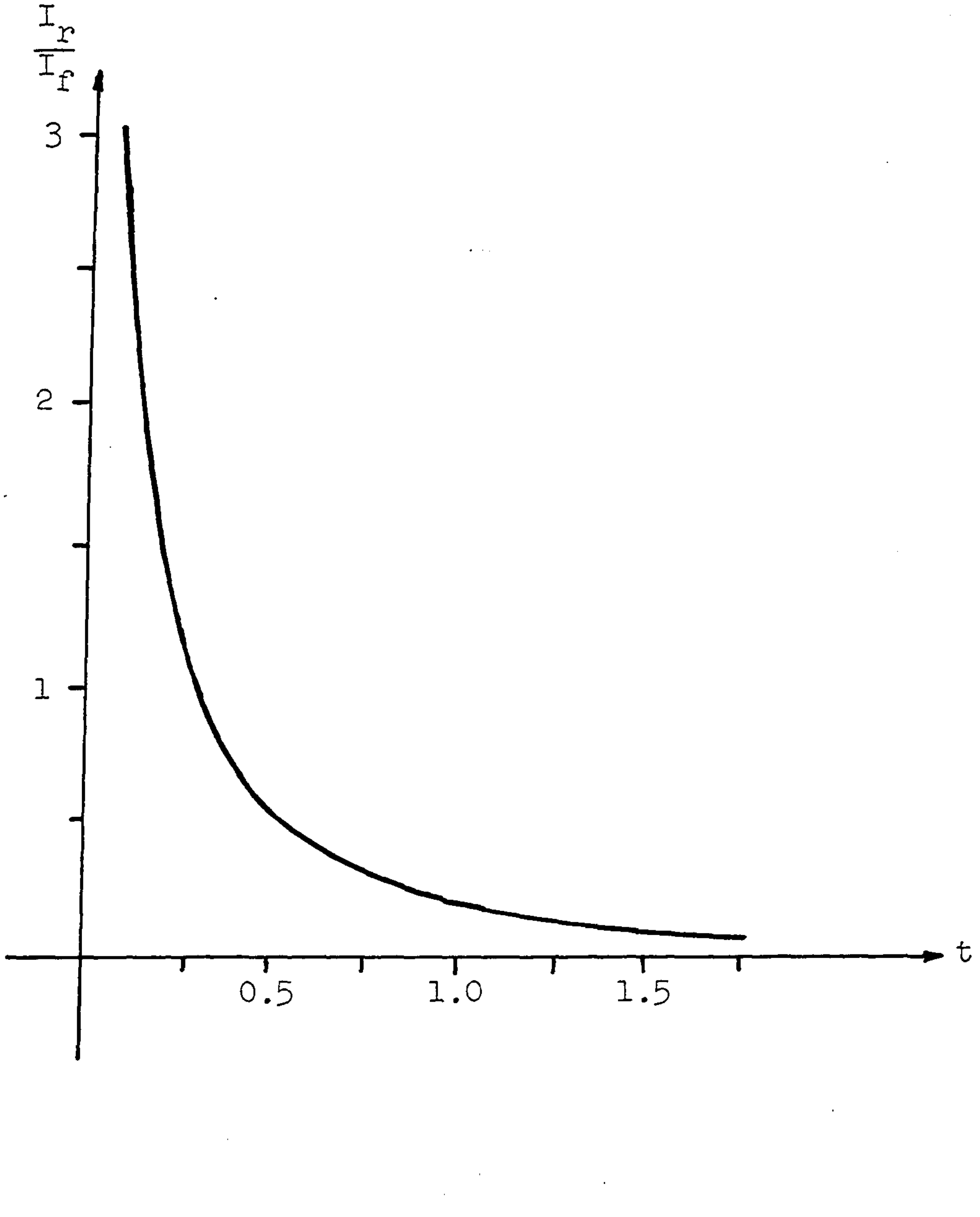
It is of interest to compare this result given by Equation 4-7 to the result obtained for the uniform base p-n junction diode. The expression for the storage time in the p-n junction diode has been derived in the literature (6), and is given by Equation 4-8.

$$\operatorname{erf}\sqrt{t_s} = \frac{I_f}{I_r + I_f} \qquad 4\text{-}8$$

Since the mobility ratio b is less than unity in both germanium and silicon, a comparison of the two equations indicates that the storage time will be slightly longer for the PIN junction if the bulk carrier lifetimes are the same in both diodes.

The other result to be obtained from Equations 3-33 and 3-37 is to determine the total recovery time t_r. This is

Figure 7. The storage time as a function of the forward to reverse current ratio


$\frac{I_r}{I_f}$
3
2
1
t
0.5
1.0
1.5

defined as the total time required for the diode to reach a value of reverse current that is ten per cent of the current observed during the storage time. This too, is most conveniently represented in terms of I_f and I_r.

To determine t_r, Equation 3-37, which describes the non constant recovery current, must be evaluated twice. First at time t_2 which is the time when I_r is equal to V_r/R, and then at time t_3 which is the time when I_r is equal to 0.1 V_4/R. By determining t_2 and t_3 along with t_s, the total recovery time may be determined by

$$t_r = t_s + t_3 - t_2. \qquad 4\text{-}9$$

By using Equation 3-41 along with Equations 4-1 and 4-2, t_2 and t_3 are determined.

$$-\ 0.1\ I_r = I_f\ (\frac{2}{b+1})(\text{erfc}\ \sqrt{t_3} - \frac{e^{-t_3}}{\sqrt{\pi t_3}} \qquad 4\text{-}10$$

$$-I_r = I_f\ (\frac{2}{b+1})(\text{erfc}\ \sqrt{t_2} - \frac{e^{-t_2}}{\sqrt{\pi t_2}}) \qquad 4\text{-}11$$

These times t_2 and t_3 may be graphically determined quite easily from Figure 6, which is a representation of the forward to reverse current ratio as a function of time.

DISCUSSION

As was pointed out early in Chapter 3, the method employed to determine the carrier distributions is very similar to the methods previously employed to determine the carrier distribution in the p-n junction. The primary differences were that both holes and electrons were considered and the internal electric field was accounted for in the PIN diode analysis. The constant current recovery time t_s given by Equation 4-7 bears a very close resemblance to the constant current recovery time derived for the p-n junction as given by Equation 4-8.

$$\operatorname{erf} \sqrt{t_s} = \frac{I_f \left(\frac{2}{b+1}\right)}{I_r + I_f \left(\frac{2}{b+1}\right)} \qquad 4\text{-}7$$

$$\operatorname{erf} \sqrt{t_s} = \frac{I_f}{I_f + I_r} \qquad 4\text{-}8$$

In the limit of a mobility ratio of one, these two equations become identical. This is providing of course that the intrinsic region is long so that tanh d = 1 and the forward current is due entirely to recombination. The result given by Equation 4-8 is for the semi-infinite p-n junction. Thus we may also consider the forward current in the very long p-n junction diode as due to recombination of minority carrier on the long side of the junction. Thus, given a p-n junction and a PIN junction with identical lifetimes of minority carriers, the recovery times should not be expected to differ greatly from each other since the number of stored minority carriers is identical. The initial distribution functions for the two types of junctions are different, but the mechanisms which distribute the minority carriers during forward conduction also work in reverse during the recovery time to remove the minority carriers. That is, the internal field which distributes the carriers across the entire intrinsic region during forward conduction in the PIN diode also transports these carriers back across the intrinsic region during the recovery period. In the p-n junction diode, where the field effect is negligible (at least at low forward conduction), the minority carriers are transported only by diffusion during both forward conduction and the recovery period. Therefore, the recovery times should not be expected to differ greatly from each other providing that the mobilities of holes and electrons do not differ greatly from each other within the material.

The equations and results derived in this thesis should provide a good description of the reverse transient response in the PIN diode providing that the limitations introduced upon the diode are adhered to.

1. The intrinsic region is appreciably longer than a diffusion length so that nearly all forward current is due to recombination.

2. The intrinsic region is not much longer than six

diffusion lengths so that the carrier concentrations in the middle of the intrinsic region do not become excessively small.

3. Both the p and n regions must possess doping levels sufficiently-high so that $p(d, 0)$ and $n(d, 0)$ are much less than the impurity levels in the p and n regions.

4. The constant recovery current should not be orders of magnitude greater than the forward conduction current because this would proportionally increase the error in the assumption that $p = n$.

5. The impurity level within the intrinsic region of an actual diode must be much less than the charge levels of the mobile carriers within the intrinsic region, at least until the termination of the constant current recovery period.

With these limitations in mind, the results in this thesis should describe the $P^+N\ N^+$ diode or the $P^+\pi\ N^+$ diode during the time when the carrier distributions within the intrinsic region exceed the impurity doping level, or more precisely, when the diode is operating in the conductivity modulation mode. The π and N in the above diode configurations indicate lightly doped P and N regions respectively.

A typical device which would fulfill the limitations of this analysis would be an epitaxially grown P^+NN^+ or $P^+\pi N^+$ high voltage rectifier with a planar geometry. The N or π regions must of course possess doping levels a few orders of magnitude lower than that in the N^+ and P^+ regions. This should assure that the assumption of near perfect carrier injection is met. Also, the epitaxial growth must be kept within the tolerance of two to ten diffusion lengths in thickness. A π region 1/2 millimeter in thickness would fulfill the thickness requirement if the material possessed approximately a 10 microsecond lifetime.

Once the device described above is obtained, the results obtained in this thesis may be compared to experimentally obtained results. An actual device possessing a π region of say 1000 ohm-cm. resistivity material would possess a fixed impurity level of approximately 10^{13} ions per cm^3. From the numerical case worked out in Appendix A, a forward conduction current of about 1/2 ampere per cm^2 would result in a minority carrier density of about 10^{16} carriers per cm^3 within the π region and therefore the π region would behave for all practical purposes as an intrinsic region.

By observing the voltage developed across the resistance R in the circuit of Figure 1b, it should be possible to observe the waveform of the diode current. The switch in the circuit would be replaced by a mercury relay to provide a repeating pattern on an oscilloscope.

Once the constant current recovery time t_s is measured, the minority carrier lifetime may be calculated by means of Equation 4-7.

$$\operatorname{erf} \sqrt{t_s} = \frac{I_f\left(\frac{2}{b+1}\right)}{I_r + I_f\left(\frac{2}{b+1}\right)} \qquad 4\text{-}7$$

Where t_s is the observed constant current recovery time in terms of normalized time $t = \frac{t_1}{\tau}$. Thus by means of Equation 4-7 and the experimentally observed recovery time t_s we may obtain a theoretical value for the lifetime of minority carriers.

The only other measurement required to compare the theoretical value of τ obtained to experiment is to obtain an independent measurement of the minority carrier lifetime within the π region. This may be performed by one of the standard techniques available for measuring lifetimes (15), such as measuring the conductivity of the region as a function of time after a charged particle bombardment and relating the gradient of the conductivity curve to lifetime. Also, if an edge of the π region of the device is exposed, the commonly used photoconductivity decay method may be used. It is quite similar to the charged particle bombardment method except that photons are used as the source of exciting minority carriers.

Similar experiments have been applied to Equation 4-8 to verify the p-n junction recovery theory with successful results (15).

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APPENDIX A

Proof of the Validity of the Assumption p = n

To determine the deviation in the hole and electron distributions given by Equation 3-29, an expression for the electric field within the intrinsic region must be derived. The problem shall be handled in the manner of a perturbation problem in which the distributions given by Equation 3-29 shall be used to derive the electric field to a first approximation. From the electric field, we may derive the difference between the two distributions and then determine whether or not it is necessary to introduce a correction term onto Equation 3-29 to determine an accurate approximation to the electric field.

For mathematical simplicity, a mobility ratio of unity shall be used. This results in the following first order approximation to n and p.

$$n(x) = p(x) = \frac{\cosh x}{\cosh d} n_i e^{\frac{q(V_a+V_d)}{2kT}} \qquad \text{A-1}$$

Under the assumption made in chapter 3 that only one type of carrier can cross each boundary, we may determine an expression for the electron current due to drift and diffusion in terms of recombination of electrons within the intrinsic region. Using the model in which electrons are injected at the point x = d, the electron current due to drift and diffusion at a point x must be proportional to the recombination rate of those electrons to the right of x.

$$i_n = \frac{qL}{\tau}\int_{-d}^{x} n(y)\,dy \qquad \text{A-2}$$

This must be equal to the sum of the drift and diffusion currents at the point x.

$$q\left[n(x)\mu_n E(x) + \frac{D_n}{L}\frac{\partial n}{\partial x}\right] = \frac{qL}{\tau}\int_{-d}^{x} n(y)\,dy \qquad \text{A-3}$$

Next, Equation A-1 is substituted into Equation A-3 and terms are cancelled.

$$\mu_n \cosh x\, E(x) + \frac{D_n}{L}\sinh x = \frac{L}{\tau}\int_{-d}^{x}\cosh y\,dy \qquad \text{A-4}$$

Integrating the right side of Equation A-4 and collecting terms results in

$$E(x) = \frac{L}{\mu_n \tau}\sinh d\,\mathrm{sech}\,x\,. \qquad \text{A-5}$$

From Maxwell's equation we have

$$\frac{\partial E}{\partial x} = \frac{q}{\varepsilon}\left(p(x) - n(x)\right)\,. \qquad \text{A-6}$$

By differentiating Equation A-5 and substituting the result into Equation A-6 results in

$$p(x) - n(x) = \frac{\varepsilon L}{q\mu_n \tau}\sinh d\,\mathrm{sech}\,x\,\tanh x\,. \qquad \text{A-7}$$

For an arbitrary case in which the parameters have approximately the following values

$$\tau \sim 10^{-5}$$

$$\mu_n \sim 10^{3}$$

$$L \sim 10^{-3}$$

$$q \sim 10^{-19}$$

$$\varepsilon \sim 10^{-12},$$

A-7 yields

$$p(x) - n(x) \leq 10^{6} \sinh d. \qquad \text{A-8}$$

From Equation 3-30, which gives the forward current in terms of the recombination rate within the intrinsic region, we may determine the order of magnitude of the charge densities in the intrinsic region.

$$I_f = \frac{2Lq}{\tau} \tanh d \; n_i \; e \; \frac{q(V_a V_d)}{2kT} \qquad \text{A-9}$$

For a small forward current of .1 amperes we obtain

$$n(x) \cong 10^{16} \frac{\cosh x}{\cosh d} \qquad \text{A-10}$$

Combining Equations A-8 and A-10 results in

$$\frac{p(x) - n(x)}{n(x)} \leq 10^{10} \sinh d \cosh d \qquad \text{A-11}$$

For the junctions in which d is less than or equal to 5

$$\frac{p - n}{n} < 10^{-6}. \qquad \text{A-12}$$

This indicates that the approximation is good to one part in 10^6 and thus the correction term is negligible.

For the recovery case, Figure 5 indicates that the center portion of the distribution curves remain approximately the same as in the forward case. However, since the direction of current flow has been reversed, there will now be an excess of electrons near x = -d and an excess of holes near x = d to create a drift field in the opposite direction. We shall now examine the constant current case in which $I_r = I_f$ and determine the limit of validity of the assumption. Since any great excess of carrier deviation across the intrinsic region must set up a large field, which can happen only if the recovery current drops, the equations derived by assuming p = n put an upper limit upon the constant current recovery time. Thus if we arbitrarily assume that the approximation is good so long as the relative error is less than 1 part in 100, we may determine from Equation A-11 that the approximation is good until the distribution function given by Equation 3-37 has dropped to .01 per cent of its initial value. By evaluating Equation 3-37 at x = d and specifying $n(d, t) = 0.10^{-4}\ n(d, 0)$, we obtain

$$\operatorname{erf} \sqrt{t} = 0.49995 \qquad \text{A-14}$$

as compared to $\operatorname{erf} \sqrt{t_s} = 0.5$ Equation 4-7. From this we obtain

$$\frac{t_s - t}{t_s} \ll 0.01 \ . \qquad \text{A-15}$$

Therefore, if it is assumed that a one per cent relative deviation of p and n is considered a good approximation, then the recovery time t_s calculated under the assumption p = n also possesses a very small relative error.

During the nonconstant current recovery phase, the distribution function continues to become smaller. However, as indicated by Figure 7, the recovery current is also becoming much smaller. Therefore the drift currents across the intrinsic region are becoming much smaller. Due to the nonlinearity of the problem, it is only practical to qualitatively examine the effect of a current deviation from that calculated in Chapter 3.

If the current observed initially is higher than that calculated by Equation 3-41, then as the total recovery time t_r is approached, the observed current must fall faster than that calculated by Equation 3-41 since there is only a finite amount of charge to be depleted from the intrinsic region.

If the current observed initially is lower than that calculated, then the external field will be increased and thus result in an increased drift field and an increase in the carrier gradients near the diffusion boundaries which will result in a current increase.

Therefore, considering both possible cases of current deviation, it appears that the calculated recovery time t_r should still be a good approximation to that observed in the actual diode.

APPENDIX B

The Steady State Forward Distribution

The differential equation and boundary conditions describing the forward steady state condition are given by Equation 3-25, 3-26, 3-27, and 3-28.

$$\frac{\partial^2 n}{\partial x^2} = n \qquad \text{B-1}$$

$$n(-d) = p(-d) = n_i \exp \frac{eV_a}{kT} \qquad \text{B-2}$$

$$n(d) = p(d) = n_i \exp \frac{eV_d}{kT} \qquad \text{B-3}$$

$$\frac{\partial n}{\partial x}\Big|_{x=-d} = -b \frac{\partial n}{\partial x}\Big|_{x=d} \qquad \text{B-4}$$

The general solution of Equation B-1 is of the form

$$n(x) = C_1 \cosh x + C_2 \sinh x. \qquad \text{B-5}$$

C_1 and C_2 are arbitrary constants which must be evaluated from the boundary conditions. Inserting Equation B-5 into the boundary equations yields

$$C_1 \cosh d - C_2 \sinh d = n_i \exp \frac{eV_a}{kT} \qquad \text{B-6}$$

$$C_1 \cosh d + C_2 \sinh d = n_i \exp \frac{eV_d}{kT} \qquad \text{B-7}$$

$$-C_1 \sinh d + C_2 \cosh d = -bC_1 \sinh d - bC_2 \cosh d. \qquad \text{B-8}$$

By solving Equations B-6, B-7, and B-8, the constants C_1 and C_2 can be determined in terms of d, n_i, and $V_a + V_d$.

$$C_1 = \frac{(b+1)\cosh d}{[(b+1)^2(\cosh d)^4 - (b-1)^2(\sinh d)^4]^{1/2}} \cdot n_i \exp \frac{q(V_a+V_d)}{2kT} \qquad \text{B-9}$$

$$C_2 = -\frac{(b-1)\sinh d}{[(b+1)^2(\cosh d)^4 - (b-1)^2(\sinh d)^4]^{1/2}} \cdot n_i \exp \frac{q(V_a+V_d)}{2kT} \qquad \text{B-10}$$

By inserting Equations B-9 and B-10 into Equation B-5, the carrier distribution is completely determined.

$$n(x) = p(x) = \frac{(b+1)\cosh d \cosh x - (b-1)\sinh d \sinh x}{[(b+1)^2(\cosh d)^4 - (b-1)^2(\sinh d)^4]^{1/2}} \cdot n_i \exp \frac{q(V_a+V_d)}{2kT} \qquad \text{B-11}$$

APPENDIX C

Constant Current Recovery Solution

The equations specifying the boundary value problem that represents the constant current phase of the recovery period are 3-23, 3-34, 3-35, and 3-36.

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} - n \qquad \text{C-1}$$

$$\left.\frac{\partial n}{\partial x}\right|_{x=d} = \frac{LI_r}{\partial bqD} = I_1 \qquad \text{C-2}$$

$$\left.\frac{\partial n}{\partial x}\right|_{x=-d} = \frac{-LI_r}{2qD} = I_2 \qquad \text{C-3}$$

$$n(x, 0) = A_1 \cosh x + A_2 \sinh x \qquad \text{C-4}$$

The first step toward obtaining the solution of Equation C-1 is to take the Laplace transform with respect to time.

$$\frac{\partial^2 N}{\partial x^2} - (1+s)\, N = -A_1 \cosh x - A_2 \sinh x \qquad \text{C-5}$$

The solution of Equation C-5 is

$$n(x, s) = C_1 e^{xw} + C_2 e^{-xw} + \frac{A_1}{s} \cosh x + \frac{A_2}{s} \sinh x \qquad \text{C-6}$$

where $w = \sqrt{1+s}$. C-7

The boundary conditions given by Equations C-2 and C-3 become

$$\left.\frac{\partial N}{\partial x}\right|_{x=d} = \frac{I_1}{s} \qquad \text{C-8}$$

$$\left.\frac{\partial N}{\partial x}\right|_{x=-d} = \frac{I_2}{s} \qquad \text{C-9}$$

in the Laplace transform domain. By using Equations C-8 and C-9 to solve for C_1 and C_2, the following expression is obtained for N(x, s).

$$N(x,\ s) = -\frac{A_1}{s}\cosh x - \frac{A_2}{s}\sinh x + \frac{(I_1 + A_3 + A_4)}{sw}\,e^{-(d-x)w}$$

$$+\ \frac{(I_2-A_3+A_4)(e^{-(3d-x)w} + e^{-(d+x)w}) - (I_1+A_3+A_4)(e^{-(5d-x)w}+e^{-(3d+x)w})}{sw(1 - e^{-4dw})} \qquad \text{C-10}$$

where

$$A_3 = A_1 \sinh d$$

$$A_4 = A_2 \cosh d$$

To obtain a form of Equation C-10 for which the inverse Laplace transform can be readily determined, it is necessary to expand the denominator terms of the form 1 - exp (-4dw) in a power series (13).

$$\frac{1}{[1 - \exp(-4dw)]} = \sum_{n=0}^{\infty} \exp(-4ndw) \qquad \text{C-11}$$

Equation C-11 is now substituted into Equation C-10 and terms are collected to yield

$$n(x,\ s) = -\frac{A_1}{s} \cosh x - \frac{A_2}{s} \sinh x + \frac{(I_1 + A_3 + A_4)}{sw} e^{-(d-x)w}$$

$$+ \sum_{n=0}^{\infty} \left[\frac{(I_1 + A_3 + A_4)}{sw} \left[e^{-[(4n+5)d-x]w} + e^{-[(4n+3)d+x]w} \right] \right.$$

$$\left. - \frac{(I_2 - A_3 + A_4)}{sw} \left[e^{-[(4n+3)d-x]w} + e^{-[(4n+1)d+x]w} \right] \right]. \qquad \text{C-12}$$

The inverse Laplace transform of Equation C-12 is given by transform pair number 825 in Campbell and Foster (14). The resulting expression for n(x, t) is

$$n(x,\ t) = A_1 \cosh x + A_2 \sinh x - \frac{(I_1+A_3+A_4)}{2} \left[e^{-(d-x)} \right.$$

$$\left. \operatorname{erfc}\left(\frac{d-x}{2\sqrt{t}} - \sqrt{t}\right) - e^{d-x} \operatorname{erfc}\left(\frac{d-x}{2\sqrt{t}} + \sqrt{t}\right) \right] + \sum_{n=0}^{\infty} \left[\frac{(I_2-A_3+A_4)}{2} \right.$$

$$\left[e^{-[(4n+3)d-x]} \operatorname{erfc}\left(\frac{(4n+3)d-x}{2\sqrt{t}} - \sqrt{t}\right) - e^{[(4n+3)d-x]} \right.$$

$$\operatorname{erfc}\left(\frac{(4n+3)d-x}{2\sqrt{t}} + \sqrt{t}\right) + e^{-[(4n+1)d+x]} \operatorname{erfc}\left(\frac{(4n+1)d+x}{2\sqrt{t}} - \sqrt{t}\right)$$

$$\left. - e^{[(4n+1)d+x} \operatorname{erfc}\left(\frac{(4n+1)d+x}{2\sqrt{t}} + \sqrt{t}\right) \right] - \frac{(I_1+A_3+A_4)}{2}$$

$$\left[e^{[(4n+5)d-x]} \operatorname{erfc}\left(\frac{(4n+5)d-x}{2\sqrt{t}} + \sqrt{t}\right) + e^{-[(4n+3)d+x]} \right.$$

$$\left. \left. \operatorname{erfc}\left(\frac{(4n+3)d+x)}{2\sqrt{t}} - \sqrt{t}\right) - e^{[(4n+3)d+x]} \operatorname{erfc}\left(\frac{(4n+3)d+x}{2\sqrt{t}} + \sqrt{t}\right) \right] \right]$$

C-13

APPENDIX D

Non Constant Current Recovery Solution

The boundary value problem describing this phase of the recovery problem is given by Equations 3-23, 3-27, 3-38, and 3-39.

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} - n \qquad \text{D-1}$$

$$n(-d, t) = 0 \qquad \text{D-2}$$

$$n(d, t) = 0 \qquad \text{D-3}$$

$$n(x, o) = A_1 \cosh x + A_2 \sinh x \qquad \text{D-4}$$

First, the Laplace transform with respect to time is taken on Equation D-1.

$$\frac{\partial^2 N}{\partial x^2} - (1 + s)\, N = -A_1 \cosh x - A_2 \sinh x \qquad \text{D-5}$$

The solution of differential Equation D-5 is

$$N(x, s) = C_1\, e^{xw} + C_2\, e^{-xw} + \frac{A_1}{s} \cosh x + \frac{A_2}{s} \sinh x. \qquad \text{D-6}$$

By applying the boundary conditions to Equation D-6, C_1 and C_2 become

$$C_1 = \frac{(A_5 + A_6)}{s} e^{-dw} - \frac{(A_5 - A_6)e^{-3dw} - (A_5 + A_6)e^{-5dw}}{s(1 - e^{-4dw})} \qquad \text{D-7}$$

$$C_2 = \frac{(A_5 - A_6)e^{-dw} - (A_5 + A_6)e^{-3dw}}{s(1 - e^{-4dw})} \qquad \text{D-8}$$

By expanding the denominator terms of the form 1 - exp(-4dw) in a power series and substituting into Equation D-6, we obtain

$$n(x, s) = \frac{(A_5+A_6)}{s} e^{-(d-x)w} - \frac{A_1}{s} \cosh x - \frac{A_2}{s} \sinh x$$

$$+ \sum_{n=0}^{\infty} \Big[-\frac{(A_5-A_6)}{s} e^{-[(4n+3)d-x]w} + \frac{(A_5+A_6)}{s} e^{-[(4n+5)d-x]w}$$

$$+ \frac{(A_5-A_6)}{s} e^{-[(4n+1)d+x]w} - \frac{(A_5+A_6)}{s} e^{-[(4n+3d+x]w}. \Big] \qquad \text{D-9}$$

The inverse Laplace transform of Equation D-9 is now found with the aid of transform pair number 819 in Campbell and Foster (14).

$$n(x,t) = A_1 \cosh x + A_2 \sinh x - \frac{(A_5+A_6)}{2} [e^{-(d-x)} \mathrm{erfc}(\frac{d-x}{2\sqrt{t}} - \sqrt{t})$$

$$-e^{d-x} \mathrm{erfc}(\frac{d-x}{2\sqrt{t}} + \sqrt{t})] - \sum_{n=0}^{\infty} \Big[- \frac{(A_5-A_6)}{2} [e^{-[(4n+3)d-x} \mathrm{erfc}(\frac{(4n+3)d-x}{2\sqrt{t}}$$

$$-\sqrt{t}) + e^{[(4n+3)d-x]} \mathrm{erfc}(\frac{(4n+3)d-x}{2\sqrt{t}} + \sqrt{t}) - e^{[(4n+1)d+x]}$$

$$\mathrm{erfc}(\frac{(4n+1)d+x}{2\sqrt{t}} - \sqrt{t}) - e^{[(4n+1)d+x} \mathrm{erfc}(\frac{(4n+1)d+x}{2\sqrt{t}} + \sqrt{t})]$$

$$\frac{(A_5+A_6)}{2}\left[e^{-[(4n+5)d-x]}\operatorname{erfc}\left(\frac{(4n+5)d-x}{2\sqrt{t}}-\sqrt{t}\right)+e^{[(4n+5)d-x}\right.$$

$$\operatorname{erfc}\left(\frac{(4n+5)d-x}{2\sqrt{t}}+\sqrt{t}\right)-e^{-[(4n+3)d+x]}\operatorname{erfc}\left(\frac{(4n+3)d+x}{2\sqrt{t}}-\sqrt{t}\right)$$

$$\left.-e^{[(4n+3)d+x]}\operatorname{erfc}\left(\frac{(4n+3)d+x}{2\sqrt{t}}+\sqrt{t}\right)\right]\Bigg] \qquad \text{D-10}$$

Since the current is of most interest during this phase of the recovery problem it is obtained from Equation D-10 by taking the first derivative with respect to x and evaluating at the boundary.

$$I_r = \frac{2qD}{L}\left[A_1 \sinh d + A_2 \cosh d - (A_5+A_6)\left(\operatorname{erfc}\sqrt{t} + \frac{e^{-t}}{\sqrt{\pi t}}\right)\right] \qquad \text{D-11}$$

The time t_2 which defines the region of validity for Equation D-11 is given by

$$\frac{V_r}{R} = \frac{2qD}{L}\left[\left[A_1 \sinh d + A_2 \cosh d - (A_5+A_6)\left(\operatorname{erfc}\sqrt{t_2} + \frac{e^{-t_2}}{\sqrt{\pi t_2}}\right)\right]\right]. \qquad \text{D-12}$$